A CLASS OF LOCAL INTERPOLATING SPLINES

Edwin Catmull
Raphael Rom

University of Utah

1. INTRODUCTION

In this paper we present a general class of splines. We shall show that some known splines are special cases of these splines. Of particular interest, however, is the subclass of these splines that is local and interpolating.

The spline will be presented in a parametric form:

$$\mathbf{F}(s)=[x_1(s),\ x_2(s),\ .\ .\ .]$$

For the purpose of the mathematics it is only necessary to consider one component, say $x(s)$, since the others are treated in the same way.

For the purpose of this paper we use the following terminology:

(i) Defining points: a set of ordered data points p_i that are evenly spaced in s. In our examples we shall usually use two dimensions.

(ii) Spline: A piecewise function with preset properties of continuity and differentiability.

(iii) Interpolating spline: a spline that passes through its defining points.

(iv) Approximating splines: a spline that may not pass

This work was supported by the Advanced Research Projects Agency of the Department of Defence under contract number DAHC15-73-C-0363.

through its defining points.

(v) Local spline: a spline that changes in a finite interval when one of its defining points is changed.

(vi) Cardinal function: a function that is 1 at some knot, 0 at all other knots and can be anything in between the other knots. It satisfies $F_i(s_j)=\delta_{ij}$.

2. THE MODEL

Consider two functions of s: $x_1(s)$ and $x_2(s)$. The average function $F(s)=(x_1(s)+x_2(s))/2$ is a function that for each s passes midway between the two given functions. We may also assign different weights and have

$$F(s)=(w_1x_1(s)+w_2x_2(s))/(w_1+w_2)$$

thus emphasizing the effect of one function over the other. This is merely a weighted average of x_1 and x_2. Finally this can be extended to make w a function of s thus varying the weight on the x's as we vary s. Also the number of functions can be increased and the model of the spline will then be

$$F(s)=\sum x_i(s)\,w_i(s)\,/\sum w_i(s) \qquad (1)$$

The $w_i(s)/\sum w_i(s)$ are often called blending functions.

It should be emphasized at this point that in the model defined by equation (1), functions are blended together rather than the defining points as in other interpolating schemes.

If $w_i(s)$ is zero outside some given interval of s then $x_i(s)$ has an effect only in that interval. In other words, $x_i(s)$ has only a local effect on $F(s)$. Note that the differentiability of $F(s)$ is determined by the minimum differentiability of $x_i(s)$ and $w_i(s)$.

Consider the following case: Let $x_i(s)$ be any function interpolating the points p_i through s_{i+k}, and let $w_i(s)$ be zero outside (s_{i-1}, s_{i+k+1}). The function $F(s)$ defined in equation (1) will thus be an interpolating function. Intuitively, this says that if all of the functions that have an effect at a point, pass through the point, then the average of the functions will pass through the point.

In general, the points p_j are pairs (x_j, y_j) and in the parametric space we can, without loss of generality, place $s_j = j$.

A polynomial of degree k that passs through k+1 given points will be used as $x(s)$. In general it will not pass through the other points. If the width of the interval in which $w_i(s)$ is non zero is less than or equal to k+2 then $x_i(s)$ will not affect $F(s)$ outside the interpolation interval. This means that $F(s)$ will be an interpolating function. On the other hand if the width of $w_i(s)$ is greater than k+2 then $x_i(s)$ will have an effect on the curve outside the interpolation interval. $F(s)$ will then be an approximating function.

One example is the B-spline where the polynomials are of degree 0 $[x_i(s) = P_i]$ and $w_i(s) = N_{i,k}(s)$ the B-spline basis function. Since $\sum N_{i,k}(s) = 1$ then $F(s) = \sum P_i N_{i,k}(s)$. For cubic B-splines the width of $N_{i,3}(s)$ is 4 which is greater than the degree of the polynomial+2. Therefore the B-spline is approximating.

3. BLENDING FUNCTIONS

Since the blending functions presented above are, as of now, completely arbitrary we impose some constraints in order to make them easier to use. We shall deal only with blending functions that are zero outside of some given interval. Also we require that $\sum w_i(s)$ does not vanish for any s. We shall normalize

$w_i(s)$ so that $\sum w_i(s)=1$ for all s. In addition, since it is most likely to choose $x_i(s)$ as polynomials which are infinitely differentiable, $F(s)$ inherits the differentiability of $w(s)$. Thus a differentiability constraint must also be imposed on these blending functions.

1. A blending function already used for approximating splines is the B-spline basis function. It has been used for blending together points (constant functions) to get an approximating spline. We have extended its use to blend functions together. There are several ways of generating the basis function [6].

2. Another function that was tried was a sort of tapered end window with more control over the differentiability (see figure 3). This is an even function that is zero for $|t| \geq t_1$ (see figure 1) and the part of the function between $t_{1/2}$ and t_1 is skew symmetric about $t_{1/2}$. This latter portion was generated using Bézier curves [3,5] for the set of points spaced as indicated in figure 1. (crosses mark the points) By virtue of

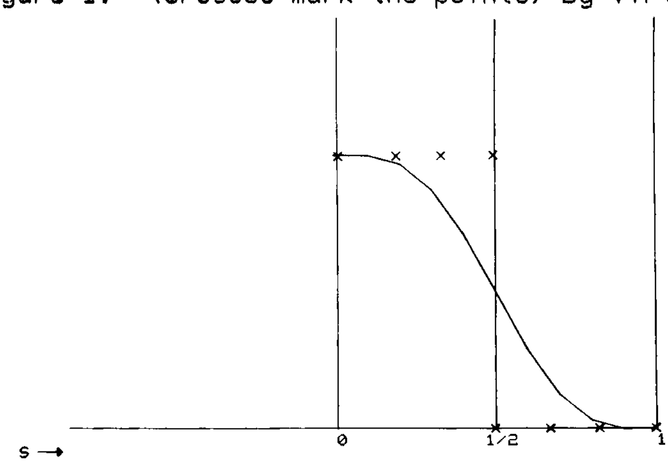


Figure 1

a property of Bézier curves, the differentiability of the function depends linearly upon the number of points. The example in Figure 1 will yield a curve of differentiability 3.

3. The previous two blending functions are piecewise polynomials. In general we might make a blending function out of pieces of polynomials where the ends of the pieces have continuity and differentiability constraints.

4. CALCULATING CARDINAL FUNCTIONS

If in equation (1) we assume $x_i(s)$ to be polynomials of degree k then this equation can be reduced to a much simpler form:

$$F(s) = \sum_j p_j C_{jk}(s) \tag{2}$$

where the $C_{jk}(s)$ are cardinal blending functions and j is the knot to which the cardinal function and the point belong and each $C_{jk}(s)$ is a shifted version of $C_{0,k}(s)$. $C_{0,k}(s)$ is a function of both the degree k of the polynomials and the blending function $w(s)$:

$$C_{0,k}(s) = \sum_{i=0}^{k} \left[\prod_{\substack{j=i-k \\ j\neq 0}}^{i} (s/j+1)\right] w(s+i) \tag{3}$$

In essence we see that for a polynomial case our cardinal functions are a blend of Lagrange polynomials. When calculating $C_{0,k}(s)$, $w(s)$ should be centered about k/2.

We have thus shown a way of creating sets of cardinal functions that are non-zero in a finite interval and the differentiability of which can be easily controlled. This result enables us to reduce the computation when creating interpolating splines.

5. EXAMPLES

To demonstrate this class of splines we have chosen to blend polynomials using both the B-spline and Bézier curves as blending functions. Our parameters are:

1. Differentiability
2. Degree of polynomials to be blended
3. The localness of the spline (which determines whether it interpolates or approximates)
4. Type of blending function (B-spline or Bézier curve)

To demonstrate the functions we are using a two dimensional case $\mathbf{F}(s)=[X(s),Y(s)]$

Figure 2 shows a B-spline blending function with differentiability 1. The vertical lines represent the knots' coordinates. Figure 3 shows a Bézier curve type blending function with differentiability 2 and width 4. We have already shown that the blending together of polynomials is equivalent to blending points with a corresponding cardinal function. If the blending function of figure 2 is to be applied to polynomials of degree 1 (i.e. the straight lines passing through adjoining points) then the corresponding cardinal function is shown in figure 4.

The blending function of figure 4 when applied to the points yield the spline of figure 5. Figure 6 shows a cardinal function made for polynomials of degree 2 using B-spline blending functions of differentiability 2. Figure 7 shows the resulting spline.

6. EXTENSIONS

By taking the cartesian cross product of two splines one can get a bivariate surface that interpolates a grid of points.

As an example, we can find the coefficients of bicubic patches that interpolate a grid of points. The cardinal function of figure 4 is a combination of the B-spline basis function of differentiability 1 and linear functions, which yields a cubic.

The formulation for a surface patch using that cardinal function can be shown to be:

$$[s^3\ s^2\ s\ 1]\ M \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix} M^T\ [t^3\ t^2\ t\ 1]^T$$

$$\text{where } M=1/2 \begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

and P_{ij} are point values. The patch interpolates the middle four points. Adjoining patches have continuity of the first derivative. This can be compared with other methods for generating bicubic patches in [1,2,4].

7. CONCLUSION

We have presented a class of splines in equation (1) that has some useful characteristics for design purposes because it is local and interpolating. We think this spline bears further investigation on its properties.

REFERENCES

[1] Bézier, P. "Mathematical and Practical Possibilities of UNISURF." These proceedings.

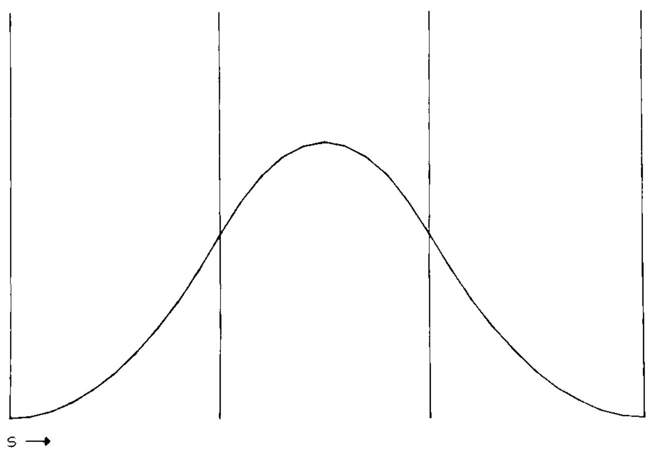
[2] Coons, S. A. "Surfaces for Computer-Aided Design of Space Forms." Project MAC, MIT, 1964. Revised to MAC-TR--41, 1967.

[3] Forrest, A. R. "Interactive Interpolation and Approximation by Bézier Polynomials." Computer Journal, Vol. 15 (1972), pp. 71-79.

[4] Forrest, A. R. "On Coons and Other Methods for the Representation of Curved Surfaces." Computer Graphics and Image Processing (1972) pp. 341-359.

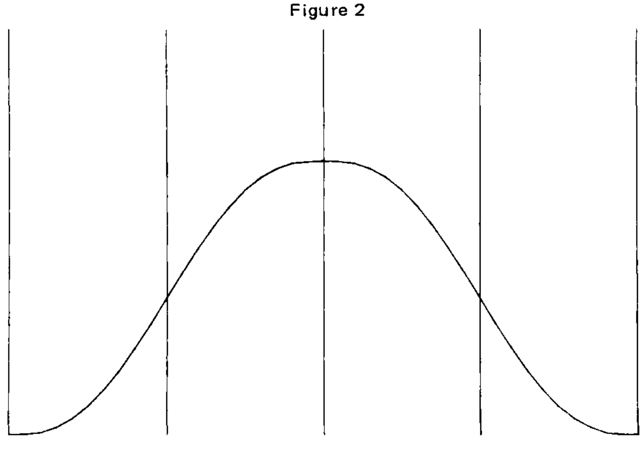
[5] Gordon, W. J. and Riesenfeld, R. F. "Bernstein-Bézier Methods for the Computer-Aided Design of Free-Form Curves and Surfaces." JACM (April 1974)

[6] Gordon, W. J. and Riesenfeld, R. F. "B-Spline Curves and Surfaces." These proceedings.



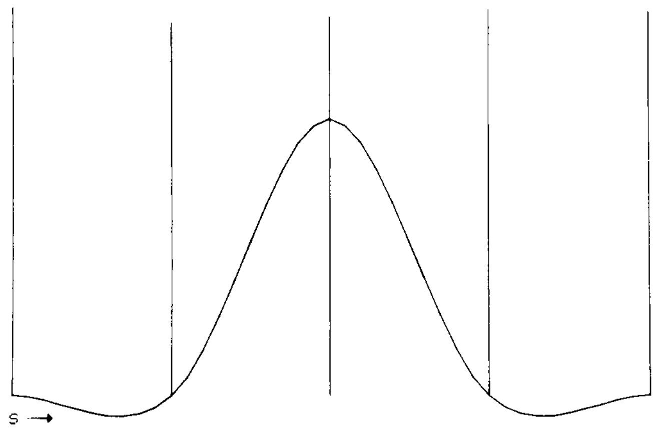
INTERVAL WIDTH=3, DIFFERENTIABILITY=1, TYPE: B-SPLINE

Figure 2



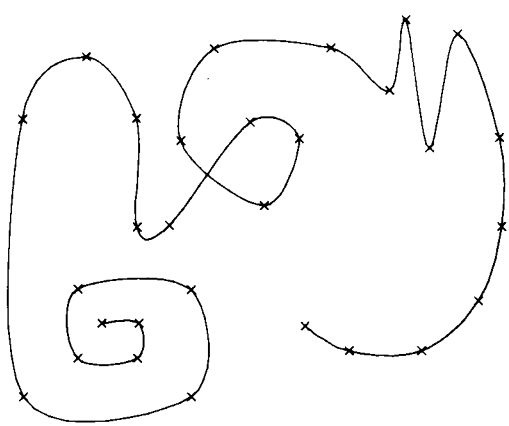
INTERVAL WIDTH=4 DIFFERENTIABILITY=2, TYPE: BEZIER

Figure 3



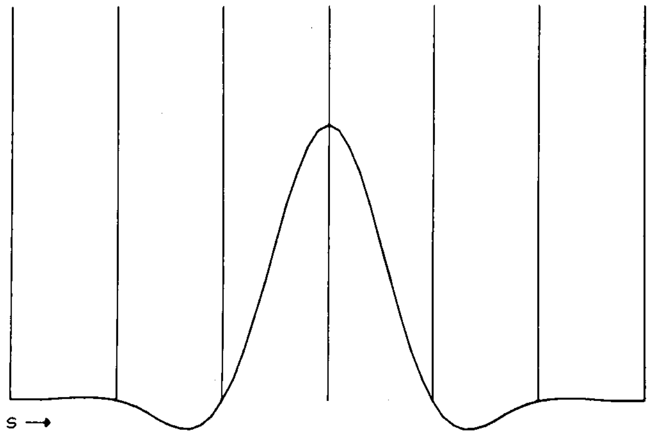
INTERVAL WIDTH=4, DIFFERENTIABILITY=1, TYPE B-SPLINE
DEGREE OF POLYNOMIAL FOR CARDINAL FUNCTION IS 1

Figure 4



DEGREE OF POLYNOMIAL FOR CARDINAL IS 1
DIFFERENTIABILITY=1, TYPE B-SPLINE
WIDTH OF BLENDING FUNCTION=4

Figure 5



INTERVAL WIDTH=6, DIFFERENTIABILITY=2, TYPE: B-SPLINE
DEGREE OF POLYNOMIAL FOR CARDINAL FUNCTION IS 2

Figure 6

DEGREE OF POLYNOMIAL FOR CARDINAL IS 2
DIFFERENTIABILITY=2, TYPE: B-SPLINE
WIDTH OF BLENDING FUNCTION=6

Figure 7